Real-rootedness conjectures in matroid theory

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Preface

The goal of this document is to introduce two interesting conjectures that June Huh and I formulated in January 2020 and to present some of the progress that has been made towards a proof of the conjecture. I assume as little background as possible: I do not assume that the reader knows what a matroid is; I do not assume that the reader has encountered real-rootedness before; the proof of Theorem 4.3.2 assumes some knowledge of toric varieties, but the proof can be skipped. The first three chapters are expository; the fourth chapter and the appendices contain original work.

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Chapter 1.

What is a matroid?

Matroids capture the general notion of “independence”.

We begin with two examples that will motivate one of many equivalent definitions[1].

Example 1.0.1. Pick a vectorspace V over some field and then choose a finite, nonempty set $E = \{v_1, \cdots, v_n\}$ of $n \leq |V|$ vectors; Now, define a subset $\mathscr{I}$ of $\wp(E)$ as follows:

$$\mathscr{I} := \{I \subset E : I \text{ is linearly independent}\}.$$

Note that the set $\mathscr{I}$ always contains the empty set. It is also closed under taking subsets, for any subset of a linearly independent set is itself linearly independent. Finally, suppose that A and B are both sets in $\mathscr{I}$ and $|A| < |B|$. Then, by the Steinitz exchange lemma, there exists a vector v that is in B but not in A such that $A \cup \{v\}$ is itself in $\mathscr{I}$. That is, linearly independent subsets of E can “steal” elements from larger linearly independent subsets and remain independent.

Example 1.0.2. Pick a field extension L/K, preferably transcendental so as to make this example interesting. Then, choose a finite nonempty set $E = \{x_1, \cdots, x_n\}$ of $n \leq |L|$ elements in L. Now, define a subset $\mathscr{I}$ of $\wp(E)$ as follows:

$$\mathscr{I} := \{I \subset E : I \text{ is } \textit{algebraically} \text{ independent}\}.$$

Here, just as above, the empty set is in $\mathscr{I}$, the set $\mathscr{I}$ is closed under taking subsets, and if A and B are sets in $\mathscr{I}$ with $|A| < |B|$, then there is an element v in $B \setminus A$ with $A \cup \{v\} \in \mathscr{I}$ by the Steinitz exchange lemma for transcendental extensions[2].

Definition 1.0.3. A **matroid** consists of a finite set E, called the **groundset**, together with a set $\mathscr{I} \subset \wp(E)$ of subsets of E that satisfies the following properties:

1. the empty set is in $\mathscr{I}$;
2. if A is an element of $\mathscr{I}$, then so are all subsets of A; and

[1] To read about other definitions and the general theory of matroids, we recommend the wonderful book by Oxley [Oxl06].

[2] If you haven’t seen a proof of this, you can read one in the proof of Theorem 1.1 in Chapter VIII of [Lan02]

3. if A and B are elements of $\mathscr{I}$ and $|A| < |B|$, then there exists an element v inside $B \setminus A$ such that $A \cup \{v\}$ is an element of $\mathscr{I}$.

Elements of $\mathscr{I}$ are called **independent sets**.

Remark 1.0.4. An *isomorphism* between two matroids is a bijection on their groundsets under which independent sets correspond to independent sets. One might then ask whether all matroids are isomorphic to those described in Examples 1.0.1 and 1.0.2. The answer is "No"; see the appendix of [Oxl06] for examples of matroids that belong to neither class and [Nel18] for a proof that almost all matroids are not isomorphic to those described in Example 1.0.1. The matroids described in Example 1.0.1 are called *linear matroids* while the matroids described in Example 1.0.2 are called *algebraic matroids.*

Example 1.0.5. For nonnegative integers r and n with $n \geq r$, the *uniform matroid* U_r^n is the matroid whose groundset is $\{1, \cdots, n\}$ and whose independent sets are all of those subsets of size at most r. Every bijection from the groundset of U_r^n to itself is an isomorphism.

Definition 1.0.6. If M is a matroid with groundset E, the **rank of a subset $S \subset E$** is the size of the largest independent set of M contained in S; it is denoted $\mathrm{rk}_M(S)$. The **rank of M** is the rank of E; it is denoted $\mathrm{rk}(M)$. A subset of E that is maximal for its rank is called a **flat** of M. The flats of M form a poset with the relation $\leq$ defined by inclusion.

Example 1.0.7. The flats of the uniform matroid U_r^n are its independent sets and the entire groundset. If S is a subset of the groundset of U_r^n, then $\mathrm{rk}_{U_r^n}(S)$ is equal to $|S|$ if $|S|$ is not more than r; otherwise it is equal to r. Note that rank and the size of the groundset uniquely determine a uniform matroid. For this reason, we will often refer to U_r^n as the "rank r uniform matroid on n elements".

Example 1.0.8. Suppose that M is a linear matroid with groundset E, so it is isomorphic to a matroid M' whose groundset E' is a set of vectors in a vectorspace V and whose independent subsets are linearly independent subsets. Let $f : E \to E'$ be an isomorphism between M and M'. If S is a subset of E, then the rank $\mathrm{rk}_M(S)$ is equal to the dimension of $\mathrm{span}_V(f(S))$. A subset S of E is a flat if and only if there holds $S = \mathrm{span}_V(f(S)) \cap E'$.

Now, we define a few useful matroid constructions.

Definition 1.0.9. Suppose that M is a matroid with groundset E and that S is a subset of E with complement T, so there hold $S \cup T = E$ and $S \cap T = \emptyset$. Then, one readily verifies that the set T together with the collection of independent sets of M that are contained in T are a matroid. This matroid is called the **deletion of S from M** or, alternatively, the **localization of M at T**, and is denoted $M \setminus S$ or M^T. The rank of M^T is the rank of T and the poset of flats of M^T can be naturally identified with the flats contained in T.

Definition 1.0.10. Suppose that M is a matroid with groundset E and that S is a subset of E. Pick a maximal (with respect to size) independent subset I of S. Then, one readiliy verifies that the set $E \setminus S$ together with the collection of subsets of $E \setminus S$ whose union with I is independent in M is a matroid and that any choice of maximal independent subset of S yields the same matroid. This matroid is called the **contraction of M by S** and is denoted M_S. The rank of M_S is $\mathrm{rk}(M) - \mathrm{rk}_M(S)$ and its poset of flats can be naturally identified with the poset flats of M that contain S.

Definition 1.0.11. Suppose that M and N are matroids with groundsets E_1 and E_2, respectively. Then, it is straightforward to verify that the set $E_1 \sqcup E_2$ together with the collection all subsets of $E_1 \sqcup E_2$ of the form $I_1 \cup I_2$ for independent subsets $I_1 \subset E_1$ and $I_2 \subset E_2$ forms a matroid. This matroid is called the **direct sum of M and N** and is denoted $M \oplus N$.

Remark 1.0.12. If M and N are matroids with groundsets E_1 and E_2 respectively, a *strong map of matroids* is a map $E_1 \to E_2$ such that preimages of flats in E_2 are flats in E_1. Matroids together with strong maps form a category. In this category, the matroid $M \oplus N$ really is the coproduct of M and N. We direct the interested reader towards the beautiful paper by Heunen and Patta [HP17].

Chapter 2.

Chow rings of matroids

For the remainder of the document, for technical reasons, every matroid will be assumed to have a set of flats that contains the empty set. This condition is called *looplessness.*

We begin by defining the *Chow ring* and *augmented Chow ring* of a matroid. The introduction of these rings was motivated by cohomology rings of compactifications of hyperplane arrangement complements; we refer the reader interested in this perspective to the wonderful paper [FY04]. Chow rings and augmented Chow rings of matroids have played key roles in the resolutions of longstanding problems in matroid theory, including the *Heron-Rota-Welsh conjecture* (see [AHK18]) and the *Downling-Wilson conjecture* (see [Bra+20b]).

Definition 2.0.1. If M is a matroid with groundset E, the quotient of the ring

$$\boldsymbol{Q}[x_F : F \text{ is a nonempty proper flat of } M].$$

by the sum of ideals

$$\left(\sum_{F \ni i_1} x_F - \sum_{F \ni i_2} x_F \text{ for all } i_1, i_2 \in E, i_1 \neq i_2\right) + \left(x_{F_1} x_{F_2} \text{ for all } F_1, F_2 \text{ incomprable nonempty proper flats of } M\right)$$

is called the **Chow ring of** M. It is denoted $\underline{\mathrm{CH}}(M)$.

Definition 2.0.2. If M is a matroid with groundset E, the quotient of the ring

$$\boldsymbol{Q}[x_F : F \text{ is a nonempty proper flat of } M].$$

by the sum of ideals

$$\left(\sum_{F \ni i_1} x_F - \sum_{F \ni i_2} x_F \text{ for all } i_1, i_2 \in E, i_1 \neq i_2\right)$$

and

$$\left(x_{F_1} x_{F_2} \text{ for all } F_1, F_2 \text{ incomparable nonempty proper flats of } M\right)$$

is called the **augmented Chow ring of** M. It is denoted $\mathrm{CH}(M)$.

Definition 2.0.3. If M is a matroid with groundset E, the quotient of the ring

$$\boldsymbol{Q}[y_i : i \in E] \otimes \boldsymbol{Q}[x_F : F \text{ is a proper flat of } M]$$

by the sum of the ideals

$$\begin{aligned} &\left(y_i - \sum_{i \notin F} x_F : i \in E\right) \\ &+ \left(x_{F_1} x_{F_2} \text{ for all } F_1, F_2 \text{ incomparable proper flats of } M\right) \\ &+ \left(y_i x_F \text{ for every } i \in E \text{ and proper flat } F \text{ with } F \not\ni i\right) \end{aligned}$$

is called the **augmented Chow ring of** M. It is denoted $\mathrm{CH}(M)$.

Theorem 2.0.4 (Proposition 2.8 in [Bra+20a]). *Suppose that M is a matroid of rank r with a nonempty groundset. Then, there exists a well-defined linear map $\underline{\deg}_M : \underline{\mathrm{CH}}^{r-1}(M) \to \boldsymbol{Q}$ such that if $\mathscr{F}$ is a complete flag of nonempty proper flats, then there holds $\underline{\deg}_M \left(\prod_{F \in \mathscr{F}} x_F\right) = 1$. Furthermore, there exists a well-defined linear map $\deg_M : \mathrm{CH}^r(M) \to \boldsymbol{Q}$ such that if $\mathscr{F}$ is a complete flag of proper flats, then there holds $\deg_M \left(\prod_{F \in \mathscr{F}} x_F\right) = 1$.*

For every integer k, these maps define *Poincaré pairings*

$$\begin{aligned} \underline{\mathrm{CH}}^k(M) \times \underline{\mathrm{CH}}^{d-k-1}(M) \to \boldsymbol{Q} : (\eta_1, \eta_2) \mapsto \underline{\deg}_M(\eta_1\eta_2), \\ \mathrm{CH}^k(M) \times \mathrm{CH}^{d-k}(M) \to \boldsymbol{Q} : (\eta_1, \eta_2) \mapsto \deg_M(\eta_1\eta_2). \end{aligned}$$

Theorem 2.0.5 (Poincaré duality, Theorem 1.3 in [Bra+20a]). *Let M be a matroid of rank r with a nonempty groundset. For every integer $0 \le k < \frac{r}{2}$, the Poincaré pairing*

$$\underline{\mathrm{CH}}^k(M) \times \underline{\mathrm{CH}}^{d-k-1}(M) \to \boldsymbol{Q} : (\eta_1, \eta_2) \mapsto \underline{\deg}_M(\eta_1\eta_2)$$

is nondegenerate. Furthermore, for every integer $0 \le i \le \frac{r}{2}$, the Poincaré pairing

$$\mathrm{CH}^k(M) \times \mathrm{CH}^{d-k}(M) \to \boldsymbol{Q} : (\eta_1, \eta_2) \mapsto \deg_M(\eta_1\eta_2)$$

is nondegenerate.

Braden, Huh, Matherne, Proudfoot, and Wang prove the following remarkable decomposition theorems for $\underline{\mathrm{CH}}(M)$ and $\mathrm{CH}(M)$.

Theorem 2.0.6 (Semi-small decomposition, Theorem 1.1 in [Bra+20a]). *Let M be a matroid with groundset E and poset of flats $\mathscr{F}$. If i is an element of E, let $\underline{\mathscr{S}}_i(M)$ denote the set $\{F \in \mathscr{F} : F \cup \{i\} \in \mathscr{F} \text{ and } \emptyset \subsetneq F \subsetneq E \setminus \{i\}\}$. If i is not contained in every basis of M, then there holds*

$$\underline{\mathrm{CH}}(M) \cong \underline{\mathrm{CH}}(M \setminus \{i\}) \oplus \bigoplus_{F \in \underline{\mathscr{S}}_i} \underline{\mathrm{CH}}(M_{F \cup \{i\}}) \otimes \underline{\mathrm{CH}}(M^F)[-1].$$

If i is contained in every basis of M, then there holds

$$\underline{\mathrm{CH}}(M) \cong \underline{\mathrm{CH}}(M \setminus \{i\}) \oplus \underline{\mathrm{CH}}(M \setminus \{i\})[-1] \oplus \bigoplus_{F \in \underline{\mathscr{S}}_i} \underline{\mathrm{CH}}(M_{F \cup \{i\}}) \otimes \underline{\mathrm{CH}}(M^F)[-1].$$

Theorem 2.0.7 (Theorem 1.2 in [Bra+20a]). *Let M be a matroid with groundset E and poset of flats $\mathscr{F}$. If i is an element of E, let $\mathscr{S}_i(M)$ denote the set $\{F \in \mathscr{F} : F \cup \{i\} \in \mathscr{F}$ and $F \subsetneq E \setminus \{i\}\}$. If i is not contained in every basis of M, then there holds*

$$\mathrm{CH}(M) \cong \mathrm{CH}(M \setminus \{i\}) \oplus \bigoplus_{F \in \mathscr{S}_i} \underline{\mathrm{CH}}(M_{F\cup\{i\}}) \otimes \mathrm{CH}(M^F)[-1].$$

If i is contained in every basis of M, then there holds

$$\mathrm{CH}(M) \cong \mathrm{CH}(M \setminus \{i\}) \oplus \mathrm{CH}(M \setminus \{i\})[-1] \oplus \bigoplus_{F \in \mathscr{S}_i} \underline{\mathrm{CH}}(M_{F\cup\{i\}}) \otimes \mathrm{CH}(M^F)[-1].$$

These decompositions give a method for computing the Poincaré polynomials of these rings.

Corollary 2.0.8. *Suppose that M is a matroid. Then, the respective Poincaré polynomials $\underline{\mathrm{P}}(M,x)$ and $\mathrm{P}(M,x)$ of $\underline{\mathrm{CH}}(M)$ and $\mathrm{CH}(M)$ can be computed as follows. Let i be an element of M and let the sets $\underline{\mathscr{S}}_i$ and $\mathscr{S}_i$ be defined as in Theorems 2.0.6 and 2.0.7. If i contained in every basis of M and the groundset of M has at least two elements, then there hold*

$$\underline{\mathrm{P}}(M,x) = \underline{\mathrm{P}}(M \setminus \{i\},x) + x \sum_{\underline{\mathscr{S}}_i} \underline{\mathrm{P}}(M_{F\cup\{i\}},x)\, \underline{\mathrm{P}}(M^F,x)$$

and

$$\mathrm{P}(M,x) = \mathrm{P}(M \setminus \{i\},x) + x \sum_{\mathscr{S}_i} \underline{\mathrm{P}}(M_{F\cup\{i\}},x)\, \mathrm{P}(M^F,x).$$

If i is not contained in every basis of M and the groundset of M has at least two elements, then there hold

$$\underline{\mathrm{P}}(M,x) = \underline{\mathrm{P}}(M \setminus \{i\},x) + x\,\underline{\mathrm{P}}(M \setminus \{i\},x) + x \sum_{\underline{\mathscr{S}}_i} \underline{\mathrm{P}}(M_{F\cup\{i\}},x)\, \underline{\mathrm{P}}(M^F,x)$$

and

$$\mathrm{P}(M,x) = \mathrm{P}(M \setminus \{i\},x) + x\,\mathrm{P}(M \setminus \{i\},x) + x \sum_{\mathscr{S}_i} \underline{\mathrm{P}}(M_{F\cup\{i\}},x)\, \mathrm{P}(M^F,x).$$

If M consists of a single element, then $\underline{\mathrm{P}}(M,x) = 1$ and $\mathrm{P}(M,x) = x + 1$.

Chapter 3.

Real-rootedness

In this chapter, we explore the basic theory of polynomials in $\boldsymbol{R}[x]$ that have all their roots on the real line. We start by exploring the interesting combinatorial properties of such polynomials, then explore the illuminating example of the Eulerian polynomials, which will turn out to be directly relevant in our discussion of the geometry of matroids.

3.1. Definition and basic properties

Definition 3.1.1. A polynomial in $\boldsymbol{R}[x]$ is called **real-rooted** if all of its roots in $\boldsymbol{C}$ lie on the real line. A subset $S \subset \boldsymbol{R}[x]$ is called **real-rooted** if each of its elements is real-rooted.

The following notion will be useful for proving real-rootedness.

Definition 3.1.2. Suppose that f and g are two nonzero, real-rooted polynomials with degrees n and $n+1$, respectively. Let $r_1 \leq \cdots \leq r_n$ denote the roots of f and let $s_1 \leq \cdots \leq s_{n+1}$ denote the roots of g. If there holds $s_1 \leq r_1 \leq s_2 \leq \cdots \leq r_n \leq s_{n+1}$, then f is said to **interlace** the polynomial g.

The "coefficient functions" give a bijection between $\boldsymbol{R}[x]$ and the topological space $\boldsymbol{R}^\infty$. This allows us to endow $\boldsymbol{R}[x]$ with a topological structure. With this structure, the subset of polynomials with simple (complex) roots is dense in $\boldsymbol{R}[x]$, by the continuity of roots. The continuity of roots also yields the following useful lemma.

Lemma 3.1.3. *If $(f_i)_{i=1}^\infty$ is a sequence of real-rooted polynomials, then $\lim_i f_i$ is real-rooted. If $(g_i)_{i=1}^\infty$ is a sequence of real-rooted polynomials and f_j interlaces g_j for all j, then $\lim_i f_i$ interlaces $\lim_i g_i$.*

Lemma 3.1.4. *Suppose that $f \in \boldsymbol{R}[x]$ is a real-rooted polynomial. Then, the polynomials $\partial_x f$ and $x^{\deg f} f(\frac{1}{x})$ are also real-rooted. Furthermore, the polynomial $\partial_x f$ interlaces f.*

Proof. The real-rootedness of $\partial_x f$ and its interlacing f is clear in the case in which f (and therefore $\partial_x f$) has only simple roots. The general case follows from the density of the space of polynomials having only simple roots and Lemma 3.1.3. To see that $x^{\deg f} f(\frac{1}{x})$ is real-rooted, note that the real-rootedness of a real polynomial is equivalent to its not having roots in the complex upper half-plane. Thus, the claim follows from the fact that $z \mapsto \frac{1}{z}$ preserves the complex upper half-plane. □

The coefficients of real-rooted polynomials satisfy a number of interesting combinatorial properties. For example, we'll soon prove that if a polynomial $\sum_{i=1}^{n} a_i x^i$ is real-rooted and a_i is nonnegative for all $1 \le i \le n$, then the sequence $(a_i)_{i=1}^{n}$ satisfies two properties, *log-concavity* and *unimodality*, which we now define.

Definition 3.1.5. A finite sequence $(c_i)_{i=1}^{n}$ of real numbers is called **log-concave** if there holds $c_i^2 \ge c_{i-1}c_{i+1}$ for all $1 \le i \le n$.

Definition 3.1.6. A finite sequence $(c_i)_{i=1}^{n}$ of real numbers is called **unimodal** if there exists $0 \le \ell \le n$ such that the subsequence $(c_i)_{i=1}^{\ell}$ is non-decreasing and the subsequence $(c_i)_{i=\ell}^{n}$ is non-increasing.

Theorem 3.1.7. *Suppose that $(c_i)_{i=0}^{n}$ is a positive, log-concave sequence. Then $(c_i)_{i=0}^{n}$ is unimodal.*

Proof. Since our sequence is positive, we can write $c_i/c_{i+1} \ge c_{i-1}/c_i$, so the sequence of successive quotients of terms is non-increasing. □

Lemma 3.1.8. *Suppose that $(c_i)_{i=1}^{n}$ and $(d_i)_{i=1}$ are log-concave sequences and that $c_i \ge 0$ and $d_i \ge 0$ for all $1 \le i \le n$. Then, the sequence $(c_i d_i)_{i=1}^{n}$ is log-concave.*

Proof. Obvious. □

Proposition 3.1.9. *For any fixed, positive integer n, the sequence $(\binom{n}{k})_{k=0}^{n}$ is log-concave.*

Proof. We have

$$\frac{\binom{n}{k}^2}{\binom{n}{k-1}\binom{n}{k+1}} = \frac{(k+1)(n-k+1)}{k(n-k)} > 1,$$

which proves the claim. □

Theorem 3.1.10. *Suppose that $f(x) = c_0 + c_1 x + \cdots + c_n x^n$ is a real-rooted polynomial in $\boldsymbol{R}[x]$ and that c_i is nonnegative for all $1 \le i \le n$. Then, the sequence $c_0, \cdots, c_n$ is log-concave.*

Proof. If we can show that $(c_i/\binom{n}{i})_{i=0}^{n}$ is log-concave, then Lemma 3.1.8 will imply that the sequence $(c_i/\binom{n}{i} \cdot \binom{n}{i})_{i=0}^{n}$ is log-concave, because Proposition 3.1.9 shows that $(\binom{n}{i})_{i=0}^{n}$ is log-concave. Thus, it suffices to show that the sequence $(c_i/\binom{n}{i})_{i=0}^{n}$ is log-concave.

Fix i with $1 \le i \le n-1$. Then, the polynomial $g(x) := (x^{n-i+1}(f)^{(i-1)}\left(\frac{1}{x}\right))^{(n-i-1)}$ is equal to $\frac{(n-i-1)!}{2}(i-1)!c_{i-1}x^2 + (n-i)!i!c_i x + (n-i-1)!\frac{i+1}{2}c_{i+1}$. One readily verifies that there holds

$$\operatorname{disc}(g) = n!^2\left(\left(\frac{c_i}{\binom{n}{i}}\right)^2 - \frac{c_{i-1}}{\binom{n}{i-1}}\frac{c_{i+1}}{\binom{n}{i+1}}\right).$$

But Lemma 3.1.4 shows that g is real-rooted, and thus the discriminant $\operatorname{disc}(g)$ must be nonnegative. Thus, we have

$$\left(\frac{c_i}{\binom{n}{i}}\right)^2 \geq \frac{c_{i-1}}{\binom{n}{i-1}} \frac{c_{i+1}}{\binom{n}{i+1}},$$

as desired. ◻

Corollary 3.1.11. *If f is a polynomial that is real-rooted and has positive coefficients, then the coefficients of f form a unimodal sequence.*

Proof. Combine Theorem 3.1.10 and Theorem 3.1.7. ◻

3.2. The Eulerian Polynomials

In this section, we will explore the *Eulerian polynomials*, a certain family of polynomials that consists entirely of real-rooted polynomials (we will prove this!).

Let $\mathfrak{S}_n$ denote the symmetric group of permutations on $\{1, \cdots, n\}$. Note that a permutation $\sigma \in \mathfrak{S}_n$ can be specified by the sequence $(\sigma(1), \cdots, \sigma(n))$ and every sequence of length n that contains every integer between 1 and n specifies a permutation.

If $\sigma \in \mathfrak{S}_n$ is a permutation, we denote by $\operatorname{des}(\sigma)$ the quantity $|\{i : \sigma(i) > \sigma(i+1)\}|$.

Definition 3.2.1. The **Eulerian number** $\boldsymbol{\left\langle {n \atop k} \right\rangle}$ is the quantity $|\{\sigma \in \mathfrak{S}_n : \operatorname{des}(\sigma) = k\}|$. The **$n$th Eulerian polynomial $A_n(x)$** is the polynomial $\sum_{\sigma \in \mathfrak{S}_n} t^{\operatorname{des}(\sigma)} = \sum_{i=0}^{n-1} \left\langle {n \atop k} \right\rangle t^k$.

Theorem 3.2.2. *For all $n > 0$ and $0 \leq k \leq n-1$, there holds*

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-1-k} \right\rangle.$$

Proof. Let $\tau \in \mathfrak{S}_n$ denote the permutation $k \mapsto n+1-k$ for any $1 \leq k \leq n$. Then, the map $\mathfrak{S}_n \to \mathfrak{S}_n : \sigma \mapsto \tau \circ \sigma$ is a bijection that sends the set $\{\sigma \in \mathfrak{S}_n : \operatorname{des}(\sigma) = k\}$ to the set $\{\sigma \in \mathfrak{S}_n : \operatorname{des}(\sigma) = n-1-k\}$ for any $0 \leq k \leq n-1$. ◻

Corollary 3.2.3. *The Eulerian polynomials are* palindromic*, i.e. for any n, the Eulerian polynomial $A_n(x)$ satisfies $x^{n-1} A_n\left(\frac{1}{x}\right)$.*

Now, we prove that the Eulerian numbers satisfy an interesting recurrence relation.

Lemma 3.2.4. *For any $n > 1$ and $1 < k \leq n$, there holds*

$$\left\langle {n \atop k} \right\rangle = (n-k) \left\langle {n-1 \atop k-1} \right\rangle + (k+1) \left\langle {n-1 \atop k} \right\rangle.$$

Proof. Let σ be a permutation in $\mathfrak{S}_n$ with $\operatorname{des}(\sigma) = k$. Deleting n from the sequence representing σ yields a permutation γ in $\mathfrak{S}_{n-1}$ with $\operatorname{des}(\gamma) = k$ or $\operatorname{des}(\gamma) = k-1$. Thus, every element of $\{\sigma \in \mathfrak{S}_n : \operatorname{des}(\sigma) = k\}$ arises from inserting n into the sequence associated with a permutation $\gamma \in \mathfrak{S}_{n-1}$ with $\operatorname{des}(\gamma) = k$ or $\operatorname{des}(\gamma) = k-1$.

If $\gamma \in \mathfrak{S}_{n-1}$ is such that $\operatorname{des}(\gamma) = k$, then a permutation in $\sigma \in \mathfrak{S}_n$ with $\operatorname{des}(\sigma) = k$ is obtained by inserting n in the last position in the sequence representing γ or in between two consecutive elements $\gamma(i)$ and $\gamma(i+1)$ with $\gamma(i) > \gamma(i+1)$. There are precisely $(k+1)$ ways to do this.

If $\gamma \in \mathfrak{S}_{n-1}$ is such that $\operatorname{des}(\gamma) = k-1$, then a permutation in $\sigma \in \mathfrak{S}_n$ with $\operatorname{des}(\sigma) = k$ is obtained by inserting n in the first position in the sequence representing γ or in between two consecutive elements $\gamma(i)$ and $\gamma(i+1)$ with $\gamma(i) < \gamma(i+1)$. There are precisely $n-k$ ways to do this. □

Lemma 3.2.5. *For any $n \geq 0$, the Eulerian polynomials $A_n(x)$ satisfy the following relation:*

$$A_{n+1}(x) = (1+nx)A_n(x) + x(1-x)A_n'(t).$$

Proof. We compute

$$\begin{aligned}
(1+nx)A_n(x) + x(1-x)A_n'(t) &= \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k + \sum_{k=1}^{n} n \left\langle {n \atop k-1} \right\rangle x^k \\
&\quad + \sum_{k=1}^{n-1} k \left\langle {n \atop k} \right\rangle x^k - \sum_{k=1}^{n-1} (k-1) \left\langle {n \atop k-1} \right\rangle x^k \\
&= \sum_{k=0}^{n} \left((n+1-k) \left\langle {n \atop k-1} \right\rangle + (k+1) \left\langle {n \atop k} \right\rangle \right) t^k \\
&= \sum_{k=0}^{n} \left\langle {n+1 \atop k} \right\rangle x^k \\
&= A_{n+1}(x)
\end{aligned} \tag{3.1}$$

where the second to last step follows from Lemma 3.2.4. □

Theorem 3.2.6. *The Eulerian polynomials have only simple, real roots and the nth Eulerian polynomial interlaces the $(n+1)$st Eulerian polynomial.*

Proof. We proceed via induction. The base case is easy. Suppose that the claim holds for the first n Eulerian polynomials. By Lemma 3.2.5, we may write $A_{n+1}(x) = (1+nx)A_n(x) + x(1-x)A_n'(t)$. Rearranging, we obtain

$$A_{n+1}(x) = (1+n)(xA_n(x)) + (1-x)(xA_n(x))'. \tag{$*$}$$

By Lemma 3.1.4, the polynomial $(xA_n(x))'$ interlaces the polynomial $xA_n(x)$. Thus, the sign of $(1-x)(xA_n(x))'$ alternates at the roots of $xA_n(x)$ and from $(*)$, the same is true of $A_{n+1}(x)$. So the polynomial $A_{n+1}(x)$ has roots $r_1, \cdots, r_n$ such that if A_n has roots $s_1, \cdots, s_n$, then there holds

$$s_1 < r_1 < s_2 < \cdots < s_n < r_n < 0.$$

To prove the claim, we must show that A_{n+1} has an additional root r_0 such that there holds $r_0 < s_1$. Analyzing the behavior of $A_{n+1}(x)$ and $A_n(x)$, as x tends to $-\infty$, it is straightforward to see that if n is even (resp. odd), then $A_{n+1}(x)$ is negative (resp. positive) at s_1, but that $A_{n+1}(x)$ tends to ∞ (resp. $-\infty$) as x tends to $-\infty$. This establishes the existence of r_0. $\square$

Chapter 4.

Two real-rootedness conjectures

4.1. Motivation and statement of the conjecture in the standard case

Proposition 4.1.1. *For all $n \geq 0$, the Poincaré polynomials $\underline{\mathrm{P}}(U_n^n, x)$ of $\underline{\mathrm{CH}}(U_n^n)$ is the nth Eulerian polynomial $A_n(x)$.*

Proof. Theorem 1.1 in [Bra+20a] shows that the Poincaré polynomials satisfy the recurrence relation

$$\underline{\mathrm{P}}(U_n^n, x) = \underline{\mathrm{P}}(U_{n-1}^{n-1}, x) + x \sum_{k=0}^{n-2} \binom{n-1}{k} \underline{\mathrm{P}}(U_k^k, x)\, \underline{\mathrm{P}}(U_{d-k-1}^{d-k-1}), \qquad \underline{\mathrm{P}}(U_0^0) = 1.$$

Theorem 1.5 in [Pet15] shows that these are the Eulerian numbers. □

Remark 4.1.2. Recall that the Eulerian polynomials are palindromic. This reflects the fact that the Chow rings $\underline{\mathrm{CH}}(U_n^n)$ satisfy Poincaré duality!

Recall that the Eulerian numbers are real-rooted and that the nth Eulerian polynomial interlaces the $(n+1)$st Eulerian polynomial.

Based on computational evidence, we conjecture that this generalizes as follows.

Conjecture 4.1.3. *Let M be a matroid with a nonempty groundset. Then, the polynomial $\underline{\mathrm{P}}(M)$ is real-rooted. If i is any element in the groundset of M, then the polynomial $\underline{\mathrm{P}}(M_{\{i\}})$ interlaces $\underline{\mathrm{P}}(M)$.*

4.2. Progress on the conjecture in the standard case

The real-rootedness of uniform matroids of small rank can be verified via computational methods.

Denote by c_i^r the function $\mathbf{Z}_{\geq r} \to \mathbf{Z}_{\geq 0}$ taking $n \in \mathbf{Z}_{\geq r}$ to the ith coefficient of $\underline{\mathrm{P}}(U_r^n)$.

Lemma 4.2.1. *For $r > 0$ (resp. $r = 0$) and every i, the function c_i^r is a polynomial of degree less than r (resp. degree zero).*

Proof. We proceed by induction on r. First, if r is less than 3, then Poincaré duality shows that $\underline{\mathrm{P}}(U_r^n)$ is equal to 1 or $x+1$ for all n; in other words, the functions c_i^r are constant for all i, so the claim is obvious this case.

Now, assuming the claim is true for all r up to some fixed integer s, we would like to show that the claim holds for $r = s+1$. Corollary 2.0.8 shows that for any $n > s+1$, there holds

$$c_i^{s+1}(n) - c_i^{s+1}(n-1) = \sum_{k=1}^{s-1} \sum_{\substack{j+l=i-1 \\ j\geq 0 \\ \ell \geq 0}} \binom{n-1}{k} c_j^k(k) c_\ell^{s+1-(k+1)}(n-(k+1)).$$

By the inductive hypothesis, the function $n \mapsto c_\ell^{s+1-(k+1)}(n-(k+1))$ is a polynomial of degree not greater than $s+1-k-2$. Furthermore, the function $n \mapsto \binom{n-1}{k}$ is a degree k polynomial in n. Thus, the RHS is a polynomial of degree not greater than $s-1$. From Newton's forward difference formula, it follows that c_i^{s+1} is a polynomial of degree not greater than s. □

Thus, we have shown for any fixed rank r that there exists a polynomial $F_r \in \boldsymbol{R}[x, y]$ such that for any nonnegative $y \in \mathbf{Z}$, the polynomial $F_r(x, y)$ coincides with $\underline{\mathrm{P}}(U_r^{r+y})$. The degree bound given in Lemma 4.2.1 allows us to effectively compute F_r via polynomial interpolation from the polynomials $\underline{\mathrm{P}}(U_r^{r+y})$ for $y = 0, \cdots, r$ (see Appendix B).

For uniform matroids, Conjecture 4.1.3 can be restated as follows.

Conjecture 4.2.2. *For any integer r and any integer $y > 0$, the polynomial $F_r(x, y)$ is real-rooted. Furthermore, the polynomial $x \mapsto F_r(x, y)$ interlaces the polynomial $x \mapsto F_{r+1}(x, y)$.*

Of course, we obtain a strictly more general statement by allowing y to be an *arbitrary* nonnegative real number (as opposed to an integer). For a fixed, small rank of r this statement is checkable by a computer: Since it is already known that the polynomials $\underline{\mathrm{P}}(U_n^n)$ are real-rooted, the real-rootedness portion of Conjecture 4.2.2 amounts to checking that the polynomial $y \mapsto \mathrm{Disc}_x(F_r(x, y))$ has no zeros in the nonnegative orthant, while the interlacing part amounts to checking that $y \mapsto \mathrm{res}(F_r(x, y), F_{r+1}(x, y))$ has no zeros in the nonnegative orthant.

This approach yields the following theorem.

Theorem 4.2.3. *For $r = 0, \cdots, 10$ and $n \geq r$, the polynomial $\underline{\mathrm{P}}(U_r^n)$ is real-rooted.*

Proof. Using [Wol] and [The20], we compute via interpolation the polynomial $y \mapsto \mathrm{Disc}_x(F_r(x, y))$ for $r = 0, \cdots, 15$ and verify that it has no zeros in the nonnegative orthant. Since the polynomial $\underline{\mathrm{P}}(U_r^r)$ is real-rooted for every $r \geq 0$, this yields the claim. □

It should be noted that the computation here is relatively efficient—computing $y \mapsto \mathrm{Disc}_x(F_r(x, y))$ consists mainly of computing determinants, at which the problem is reduced to checking that a univariate polynomial has no nonnegative roots, which can be done using Sturm's Theorem—and, in principle, this method should be applicable to matroids of larger rank. These computations were not performed due to practical limitations in our computing setup not directly related to the algorithm itself; a more effective implementation is in progress.

4.3. Motivation and statement of the conjecture in the augmented case

The Poincaré polynomials $\mathrm{P}(U_n^n)$ of the augmented Chow rings of the rank n uniform matroids U_n^n on n elements belong to a family of polynomials studied by Haglund and Zhang in [HZ19] that we now describe.

Let s be an element of $\mathbf{Z}_{>0}^n$. Denote by I^s the set of elements $e \in \mathbf{Z}_{\geq 0}^n$ such that there holds $0 \leq e_i < s_i$. For $e \in I^s$, Haglund and Zhang define the statistics $\mathrm{asc}(e) := |\{i \in [0,n] : \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}}\}|$ and $\mathrm{col}(e) := |\{i \in [0,n] : \frac{e_i}{s_i} = \frac{e_{i+1}}{s_{i+1}}\}|$, with the conventions $e_0 = e_{n+1} = 0$ and $s_0 = s_{n+1} = 1$. Using these statistics, they define the polynomials

$$\tilde{E}^s(x) := \sum_{e \in I^s} (1+x)^{\mathrm{col}(e)} x^{\mathrm{asc}(e)}.$$

Theorem 4.3.1 (Theorem 1.1 in [HZ19]). *For any $s \in \mathbf{Z}_{>0}^n$, the polynomial $\tilde{E}^s(x)$ is real-rooted.*

Theorem 4.3.2. *For $n \geq 1$, the polynomial $\mathrm{P}(U_n^n, x)$ is equal to $\tilde{E}^{(2,\cdots,n)}(x)$. In particular, the polynomials $\mathrm{P}(U_n^n, x)$ are real-rooted.*

Proof. Proposition 2.10 in [Bra+20a] shows that the Chow rings of the matroids U_n^n are the Chow rings of the toric varieties arising from a family of polytopes called the *stellohedra*, so computing $\mathrm{P}(U_n^n, x)$ amounts to computing the h-polynomials of the stellohedra. Theorem 3.1 in [HZ19] shows that the h-polynomials of the stellohedra are given by the polynomials $\tilde{E}^{(2,\cdots,n)}(x)$. □

Motivated by Theorem 4.3.2 and computational evidence, we make the following conjecture.

Conjecture 4.3.3. *Let M be a matroid with a nonempty groundset. Then, the polynomial $\mathrm{P}(M)$ is real-rooted. If i is any element in the groundset of M, then the polynomial $\mathrm{P}(M_{\{i\}})$ interlaces $\mathrm{P}(M)$.*

We also make the following conjecture based on computational evidence.

Conjecture 4.3.4. *For integers r and n with $0 \leq r \leq n$, the polynomial $\mathrm{P}(U_n^r)$ is equal to $\tilde{E}^{(n-r+2,n-r+3,\cdots,n)}$.*

By Theorem 4.3.1, Conjecture 4.3.4 implies the real-rootedness part of Conjecture 4.3.3 in the uniform case.

Remark 4.3.5. One might hope that for every matroid M, there exists $s \in \mathbf{Z}_{>0}$ such that $\mathrm{P}(M,x)$ is equal to $\tilde{E}^s(x)$. But an exhaustive computer search shows that there is no such s when M is the matroid $U_3^4 \oplus U_1^1$.

4.4. Progress on the conjecture in the augmented case

An analogue of the approach used to verify real-rootedness for uniform matroids of small rank in the standard case exists in the augmented case.

Denote by d_i^r the function $\mathbf{Z}_{\geq r} \to \mathbf{Z}_{\geq 0}$ taking $n \in \mathbf{Z}_{\geq r}$ to the ith coefficient of $\mathrm{P}(U_r^n)$.

Lemma 4.4.1. *For $r > 0$ (resp. $r = 0$) and every i, the function d_i^r is a polynomial of degree less than r (resp. degree zero).*

Proof. The proof is similar to that of Lemma 4.2.1.

As in that proof, we proceed by induction on r. If r is equal to 0 or 1, then Poincaré duality shows that $\underline{\mathrm{P}}(U_r^n)$ is equal to 1 or $x+1$ for all n; in other words, the functions d_i^r are constant for all i, so the claim is obvious in this case.

Corollary 2.0.8 shows that for any $n > r$, there holds

$$d_i^r(n) - d_i^r(n-1) = \sum_{k=0}^{r-2} \sum_{\substack{j+l=i-1 \\ j \geq 0 \\ \ell \geq 0}} \binom{n-1}{k} d_j^k(k) c_\ell^{r-(k-1)}(n-(k+1)).$$

By Lemma 4.2.1, the function $n \mapsto c_\ell^{r-(k+1)}$ is a polynomial of degree not greater than $r-k-2$. Furthermore, the function $n \mapsto \binom{n-1}{k}$ is a degree k polynomial in n. Thus, the RHS is a polynomial of degree not greater than $r = 2$. From here, the claim follows again from Newton's forward difference formula. ◻

As in the standard case, this shows for any fixed rank r that there exists a polynomial $G_r \in \boldsymbol{R}[x,y]$ such that for any nonnegative $y \in \mathbf{Z}$, the polynomial $G_r(x,y)$ coincides with $\mathrm{P}(U_r^{r+y})$. The polynomials $G_r(x,y)$ can be computed via interpolation just as in the standard case (see Appendix C). We obtain the following restatement of Conjecture 4.3.3 in the uniform case.

Conjecture 4.4.2. *For any integer r and any integer $y > 0$, the polynomial $G_r(x,y)$ is real-rooted. Furthermore, the polynomial $x \mapsto G_r(x,y)$ interlaces the polynomial $x \mapsto G_{r+1}(x,y)$.*

Computing the polynomials $y \mapsto \mathrm{Disc}_x(G_r(x,y))$ and $y \mapsto \mathrm{res}(G_r(x,y), G_{r+1}(x,y))$ and checking that they have no zeros in the nonnegative orthant can again be used to test the conjecture for small values of r. We obtain in particular the following analogue of Theorem 4.2.3.

Theorem 4.4.3. *For $r = 0, \cdots, 6$ and $n \geq r$, the polynomial $\mathrm{P}(U_r^n)$ is real-rooted.*

Proof. As in the proof of Theorem 4.2.3 using [Wol] and [The20], we compute via interpolation the polynomial $y \mapsto \mathrm{Disc}_x(G_r(x,y))$ for $r = 0, \cdots, 15$ and verify that it has no zeros in the nonnegative orthant. Since the polynomial $\mathrm{P}(U_r^r)$ is real-rooted for every $r \geq 0$, this proves the claim. ◻

Appendix A.

The Kazhdan-Lusztig polynomials of matroids

Conjecture 4.3.3 and Conjecture 4.1.3 appear to be related to the conjectured real-rootedness of the *Kazhdan-Lusztig polynomials of matroids*. We briefly state the definition of these polynomials, discuss a few known results, and use a strategy similar to those used in the proofs of Theorems 4.2.3 and 4.4.3 to obtain the real-rootedness of the Kazhdan-Lusztig polynomials of uniform matroids of small rank.

For a more complete perspective on the relationship between Kazhdan-Lusztig polynomials of matroids and the (augmented) Chow rings of matroids, we refer the reader to the excellent paper [Bra+20b].

Definition A.0.1. Suppose that M is a matroid with a poset $\mathscr{F}$ of flats. Let μ denote the Möbius function on $\mathscr{F}$. Then, the **characteristic polynomial** of M is the polynomial

$$\chi_M(x) := \sum_{F \in \mathscr{F}} \mu(\emptyset, F) x^{\operatorname{rk} M - \operatorname{rk}_M F}.$$

Proposition-Definition A.0.2 (Theorem 2.2 in [EPW16]). There is a unique way to assign to each matroid M an integer polynomial $\operatorname{KL}_M(x)$ such that the following hold:

1. If $\operatorname{rk} M = 0$, then $\operatorname{KL}_M(x) = 1$;
2. if $\operatorname{rk} M$ is positive, then $\deg \operatorname{KL}_M(x)$ is less than $\frac{1}{2} \operatorname{rk} M$; and
3. for every M, letting $\mathscr{F}$ denote the poset of flats of M, there holds $x^{\operatorname{rk} M} \operatorname{KL}_M(\frac{1}{x}) = \sum_{F \in \mathscr{F}} \chi_{M_F}(x) \operatorname{KL}_{M^F}(x)$.

The polynomial $\operatorname{KL}_M(x)$ is called the **Kazhdan-Lusztig polynomial** of M.

Conjecture A.0.3 (Conjecture 3.2 and 3.4 in [GPY17]). *For every matroid M, the Kazhdan-Lusztig polynomial $\operatorname{KL}_M(x)$ is real-rooted. Suppose that i is any element of the groundset of M. If $\operatorname{rk} M$ is odd, then $\operatorname{KL}_{M_{\{i\}}}(x)$ interlaces $\operatorname{KL}_M(x)$. If $\operatorname{rk} M$ is even, then $\operatorname{KL}_M(x)$ interlaces $x \operatorname{KL}_{M_{\{i\}}}(x)$.*

The conjecture is known to hold in the following cases, among others.

Theorem A.0.4 (Theorem 3.3 and preceding remarks in [GPY17]). *The Kazhdan-Lusztig polynomials of the matroids U_n^n are identically 1. The Kazhdan-Lusztig polynomials of the matroids U_n^{n+1} are real-rooted.*

Braden and Vysogorets prove that the Kazhdan-Lusztig polynomials satisfy an interesting recurrence relation that is reminiscent of those in Corollary 2.0.8.

Theorem A.0.5 (Theorem 2.8 in [BV20]). *Let M be a matroid with groundset E such that every two-element subset of E of M is independent. Let i be any element of E. If $\operatorname{rk} M$ is odd, then let $\tau(M)$ denote the coefficient of $x^{(\operatorname{rk} M-1)/2}$ in $\mathrm{KL}(M)$; otherwise, let $\tau(M)$ denote the integer 0. Let S_i denote the set of subsets of E such that F and $F \cup \{i\}$ are both flats of M. Then, there holds*

$$\mathrm{KL}_M(x) = \mathrm{KL}_{M\setminus\{i\}}(x) - x\,\mathrm{KL}_{M_{\{i\}}}(x) + \sum_{F \in S_i} \tau(M_{F\cup\{i\}})x^{(\operatorname{rk} M - \operatorname{rk} F)/2}\,\mathrm{KL}_{M^F}(x).$$

Denote by e_i^r the function $\mathbf{Z}_{\geq r} \to \mathbf{Z}$ taking $n \in \mathbf{Z}_{\geq r}$ to the ith coefficient of $\mathrm{KL}(U_r^n)$. [1]

Lemma A.0.6. *For every $r \geq 0$ and every i, the function e_i^r is a polynomial of degree not more than r.*

Proof. The proof is similar to that of Lemma 4.2.1. As in that proof, we proceed by induction on r.

If $r = 0$, then $\mathrm{KL}(U_r^n)$ is equal to 1 for all n, so e_0^0 is identically 1.

Now, assuming the claim is true for all r up to some fixed integer s, we would like to show that the claim holds for $r = s+1$. Theorem A.0.5 shows that for any $n > s+1$, there holds

$$e_i^{s+1}(n)-e_i^{s+1}(n-1) = e_{i-1}^{r-1}(n-1)+\sum_{k=0}^{s}\binom{n}{k}\left((s-k) \bmod 2\right) e_{(s-k-1)/2}^{s-k}(s-k)\sum_j e_{j-(s+1-k)/2}^{k}(k).$$

Applying the inductive hypothesis and Newton's forward difference formula, the claim follows. □

Thus, for every integer $r \geq 0$, there exists a bivariate polynomial $H_r(x,y) \in \boldsymbol{R}[x,y]$ such that for any $y \in \mathbf{Z}$, we have $H_r(x,y) = \mathrm{KL}(U_r^{r+y})$; as before, the degree bound allows us to compute these polynomials via interpolation (see Appendix D). Checking that the polynomials $y \mapsto \mathrm{Disc}_x(H_r(x,y))$ have no roots in $\boldsymbol{R}_{\geq 1}$ for small values of r, keeping in mind that the polynomials $\mathrm{KL}(U_r^{r+1})$ are known to be real-rooted, we obtain the following theorem.

Theorem A.0.7. *For $r = 0, \cdots, 15$ and $n \geq r$, the polynomial $\mathrm{KL}(U_r^n)$ is real-rooted.*

[1] It is a highly nonobvious fact that the coefficients of the Kazhdan-Lusztig polynomials of arbitrary matroids are nonnegative, so the functions e_i^r are in fact nonnegative for all r and i; see [Bra+20b] for a proof.

Proof. As in the proof of Theorem 4.2.3 using [Wol], [The20], and [You], we compute via interpolation the polynomial $y \mapsto \mathrm{Disc}_x(H_r(x,y))$ for $r = 0, \cdots, 15$ and verify that it has no zeros in $\boldsymbol{R}_{\geq 1}$. Since the polynomials $\mathrm{KL}(U_r^{r+1})$ is real-rooted for every $r \geq 0$, this proves the claim whenever n is greater than r. The claim is immediate when n equals r, since $\mathrm{KL}(U_r^r)$ is identically 1. □

Remark A.0.8. One difference between the proof of Theorem A.0.7 and the proofs of Theorems 4.2.3 and 4.4.3 is that $y \mapsto \mathrm{Disc}_x(H_r(x,y))$ will sometimes have zeroes in the nonnegative orthant because, while $\mathrm{KL}(U_r^r)$ is identically 1, Kazhdan-Lusztig polynomials of uniform matroids are nonconstant in general.

Appendix B.

The polynomials $F_r(x, y)$

We compute the polynomials $F_r(x, y-1)$ for small values of r (we compute $F_r(x, y-1)$ because this computation is more amenable to our current software setup).

r	$\boldsymbol{F_r(x, y-1)}$
0	1
1	1
2	$x+1$
3	$1+\frac{1}{2}\left(4+3y+y^2\right)x+x^2$
4	$1+\frac{1}{6}\left(30+26y+9y^2+y^3\right)x+\frac{1}{6}\left(30+26y+9y^2+y^3\right)x^2+x^3$
5	$1+\frac{1}{24}(288+238y+83y^2+14y^3+y^4)x+\frac{1}{24}(528+642y+329y^2+78y^3+7y^4)x^2+\frac{1}{24}(288+238y+83y^2+14y^3+y^4)x^3+x^4$
6	$x^5+\frac{1}{120}x^4\left(y^5+20y^4+175y^3+880y^2+2524y+3240\right)+\frac{1}{120}x^3\left(21y^5+355y^4+2385y^3+8105y^2+14334y+11040\right)+\frac{1}{120}x^2\left(21y^5+355y^4+2385y^3+8105y^2+14334y+11040\right)+\frac{1}{120}x\left(y^5+20y^4+175y^3+880y^2+2524y+3240\right)+1$
7	$x^6+\frac{1}{720}x^5\left(y^6+27y^5+325y^4+2325y^3+10834y^2+31128y+41760\right)+\frac{1}{240}x^4\left(17y^6+401y^5+3935y^4+20835y^3+63968y^2+110524y+86160\right)+\frac{1}{720}x^3\left(161y^6+3633y^5+33485y^4+162495y^3+443714y^2+661152y+434880\right)+\frac{1}{240}x^2\left(17y^6+401y^5+3935y^4+20835y^3+63968y^2+110524y+86160\right)+\frac{1}{720}x\left(y^6+27y^5+325y^4+2325y^3+10834y^2+31128y+41760\right)+1$
8	$x^7+\frac{x^6\left(y^7+35y^6+553y^5+5285y^4+34174y^3+153440y^2+441552y+609840\right)}{5040}+\frac{x^5\left(113y^7+3528y^6+47138y^5+353640y^4+1635977y^3+4760952y^2+8227932y+6607440\right)}{5040}+\frac{x^4\left(813y^7+24248y^6+303660y^5+2078300y^4+8459787y^3+20743772y^2+28929900y+18179280\right)}{5040}+\frac{x^3\left(813y^7+24248y^6+303660y^5+2078300y^4+8459787y^3+20743772y^2+28929900y+18179280\right)}{5040}+\frac{x^2\left(113y^7+3528y^6+47138y^5+353640y^4+1635977y^3+4760952y^2+8227932y+6607440\right)}{5040}+\frac{x\left(y^7+35y^6+553y^5+5285y^4+34174y^3+153440y^2+441552y+609840\right)}{5040}+1$

9	$1+(y^8+44y^7+882y^6+10808y^5+91329y^4+559916y^3+2468108y^2+7110192y+9999360)x+(239y^8+9516y^7+165326y^6+1653624y^5+10567151y^4+44999724y^3+127161684y^2+221384496y+183052800)x^2+(3361y^8+128404y^7+2106314y^6+19439056y^5+111024809y^4+405556396y^3+937947436y^2+1278537264y+802851840)x^3+(7631y^8+287724y^7+4636534y^6+41762112y^5+230678119y^4+804930756y^3+1750679636y^2+2205082128y+1259516160)x^4+(3361y^8+128404y^7+2106314y^6+19439056y^5+111024809y^4+405556396y^3+937947436y^2+1278537264y+802851840)x^5+(239y^8+9516y^7+165326y^6+1653624y^5+10567151y^4+44999724y^3+127161684y^2+221384496y+183052800)x^6+(y^8+44y^7+882y^6+10808y^5+91329y^4+559916y^3+2468108y^2+7110192y+9999360)x^7+x^8$
10	$\frac{1}{362880}(362880x^9+x^8(y^9+63y^8+1806y^7+31374y^6+370713y^5+3153087y^4+19669544y^3+87898356y^2+256109616y+367597440)+x^7(493y^9+28782y^8+744666y^7+11262132y^6+110484885y^5+735856758y^4+3365152004y^3+10316882088y^2+19464710112y+17360179200)+x^6(12421y^9+703134y^8+17476530y^7+250742772y^6+2294432301y^5+13939291926y^4+56525551100y^3+148588339608y^2+231802373088y+165180072960)+x^5(53833y^9+3005910y^8+73422042y^7+1030138956y^6+9158075913y^5+53591263110y^4+207000239828y^3+511057067544y^2+736254362304y+475501259520)+x^4(53833y^9+3005910y^8+73422042y^7+1030138956y^6+9158075913y^5+53591263110y^4+207000239828y^3+511057067544y^2+736254362304y+475501259520)+x^3(12421y^9+703134y^8+17476530y^7+250742772y^6+2294432301y^5+13939291926y^4+56525551100y^3+148588339608y^2+231802373088y+165180072960)+x^2(493y^9+28782y^8+744666y^7+11262132y^6+110484885y^5+735856758y^4+3365152004y^3+10316882088y^2+19464710112y+17360179200)+x(y^9+63y^8+1806y^7+31374y^6+370713y^5+3153087y^4+19669544y^3+87898356y^2+256109616y+367597440)+362880)$

Appendix C.

The polynomials $G_r(x, y)$

We compute $G_r(x, y-1)$ for small values of r (here again, we compute $G_r(x, y-1)$ because this computation is more amenable to our current software setup).

r	$G_r(x, y-1)$
0	1
1	1
2	$1 + (2+y)x + x^2$
3	$1 + \frac{1}{2}(8 + 5y + y^2)x + x^2$
4	$x^4 + \frac{1}{6}x^3(y^3+9y^2+32y+48) + \frac{1}{3}x^2(2y^3+15y^2+40y+42) + \frac{1}{6}x(y^3+9y^2+32y+48) + 1$
5	$x^5 + \frac{1}{24}x^4(y^4+14y^3+83y^2+262y+384) + \frac{1}{24}x^3(11y^4+130y^3+589y^2+1262y+1152) + \frac{1}{24}x^2(11y^4+130y^3+589y^2+1262y+1152) + \frac{1}{24}x(y^4+14y^3+83y^2+262y+384) + 1$
6	$x^6 + \frac{1}{120}x^5\left(y^5+20y^4+175y^3+880y^2+2644y+3840\right) + \frac{1}{60}x^4\left(13y^5+225y^4+1575y^3+5715y^2+11132y+9720\right) + \frac{1}{60}x^3\left(33y^5+550y^4+3635y^3+12110y^2+20932y+15720\right) + \frac{1}{60}x^2\left(13y^5+225y^4+1575y^3+5715y^2+11132y+9720\right) + \frac{1}{120}x\left(y^5+20y^4+175y^3+880y^2+2644y+3840\right) + 1$

Appendix D.

The polynomials $H_r(x, y)$

We compute $H_r(x, y)$ for small values of r.

r	$H_r(x,y)$
0	1
1	1
2	1
3	$1 + \frac{1}{2}(3y + y^2)x$
4	$1 + \frac{1}{6}(20y + 9y^2 + y^3)x$
5	$\frac{1}{12}x^2\left(y^4 + 10y^3 + 29y^2 + 20y\right) + \frac{1}{24}x\left(y^4 + 14y^3 + 71y^2 + 130y\right) + 1$
6	$\frac{1}{24}x^2(y^5 + 17y^4 + 101y^3 + 235y^2 + 150y) + \frac{1}{120}x(y^5 + 20y^4 + 155y^3 + 580y^2 + 924y) + 1$
7	$\frac{1}{144}x^3\left(y^6 + 21y^5 + 163y^4 + 567y^3 + 844y^2 + 420y\right) + \frac{1}{240}x^2\left(3y^6 + 73y^5 + 685y^4 + 3035y^3 + 6032y^2 + 3612y\right) + \frac{1}{720}x\left(y^6 + 27y^5 + 295y^4 + 1665y^3 + 5104y^2 + 7308y\right) + 1$
8	$\frac{1}{720}x^3\left(3y^7 + 89y^6 + 1041y^5 + 6035y^4 + 17772y^3 + 24116y^2 + 11424y\right) + \frac{1}{720}x^2\left(2y^7 + 65y^6 + 857y^5 + 5825y^4 + 21173y^3 + 37310y^2 + 21168y\right) + \frac{x\left(y^7+35y^6+511y^5+4025y^4+18424y^3+48860y^2+64224y\right)}{5040} + 1$
9	$\frac{x^4\left(y^8+36y^7+526y^6+3996y^5+16789y^4+38304y^3+43164y^2+18144y\right)}{2880} + \frac{1}{720}x^3\left(y^8 + 39y^7 + 626y^6 + 5304y^5 + 25199y^4 + 65361y^3 + 82174y^2 + 37296y\right) + \frac{x^2\left(5y^8+208y^7+3626y^6+34216y^5+187565y^4+588952y^3+942804y^2+510624y\right)}{10080} + \frac{x\left(y^8+44y^7+826y^6+8624y^5+54649y^4+214676y^3+509004y^2+623376y\right)}{40320} + 1$
10	$\frac{x^4\left(2y^9+93y^8+1812y^7+19158y^6+118878y^5+436857y^4+911788y^3+969492y^2+393120y\right)}{8640} + \frac{x^3\left(5y^9+246y^8+5127y^7+58737y^6+400680y^5+1641549y^4+3848308y^3+4536468y^2+1982880y\right)}{15120} + \frac{x^2\left(3y^9+155y^8+3434y^7+42518y^6+320747y^5+1501115y^4+4203896y^3+6217812y^2+3233520y\right)}{40320} + \frac{x\left(y^9+54y^8+1266y^7+16884y^6+140889y^5+761166y^4+2655764y^3+5753736y^2+6636960y\right)}{362880} + 1$

11	$\frac{1}{3628800}(42x^5y(y^9+55y^8+1290y^7+16830y^6+133593y^5+663135y^4+2037260y^3+3691820y^2+3530256y+1330560)+60x^4y(5y^9+289y^8+7167y^7+99486y^6+845103y^5+4507881y^4+14888953y^3+28846424y^2+29165172y+11452320)+15x^3y(15y^9+905y^8+23616y^7+348330y^6+3181059y^5+18486825y^4+67422774y^3+145440580y^2+162220536y+68571360)+5x^2y(7y^9+439y^8+12012y^7+187926y^6+1848819y^5+11825751y^4+48926738y^3+124859324y^2+172746504y+86660640)+xy(y^9+65y^8+1860y^7+30810y^6+326613y^5+2310945y^4+11028590y^3+34967140y^2+70290936y+76998240)+3628800)$
12	$\frac{1}{39916800}(66x^5y(5y^{10}+337y^9+9870y^8+164610y^7+1720965y^6+11699961y^5+51925960y^4+147023540y^3+251136480y^2+230300352y+84395520)+165x^4y(5y^{10}+350y^9+10701y^8+187362y^7+2068659y^6+14937258y^5+70726399y^4+213961678y^3+389041116y^2+376318152y+143557920)+55x^3y(7y^{10}+507y^9+16136y^8+296262y^7+3460875y^6+26725419y^5+137010734y^4+454505748y^3+914012728y^2+971443584y+398471040)+11x^2y(y+11)^2(4y^8+211y^7+4759y^6+59881y^5+458761y^4+2182564y^3+6241116y^2+9550944y+5088960)+xy(y^{10}+77y^9+2640y^8+53130y^7+696333y^6+6230301y^5+38759930y^4+167310220y^3+489896616y^2+924118272y+967524480)+39916800)$
13	$\frac{1}{12!}(132x^6y(y^{11}+78y^{10}+2675y^9+53040y^8+672603y^7+5698134y^6+32711405y^5+126373260y^4+319387396y^3+497990688y^2+425603520y+148262400)+99x^5y(15y^{11}+1210y^{10}+43051y^9+888390y^8+11759385y^7+104241150y^6+627052593y^5+2537921450y^4+6704205500y^3+10876007640y^2+9608273856y+3432274560)+55x^4y(35y^{11}+2910y^{10}+107155y^9+2299206y^8+31808541y^7+296324946y^6+1883447041y^5+8090782674y^4+22729612844y^3+39116334264y^2+36379828224y+13519059840)+44x^3y(14y^{11}+1197y^{10}+45538y^9+1015185y^8+14692242y^7+144376011y^6+977659774y^5+4526220195y^4+13869539744y^3+26260079892y^2+26745883488y+10676128320)+6x^2y(9y^{11}+790y^{10}+31009y^9+717750y^8+10872807y^7+113024010y^6+821171307y^5+4158853490y^4+14324279684y^3+31656550200y^2+39330077184y+18557285760)+xy(y^{11}+90y^{10}+3641y^9+87450y^8+1387023y^7+15282630y^6+119753843y^5+671189310y^4+2664929476y^3+7292774280y^2+13020978816y+13096736640)+479001600)$
14	$\frac{1}{13!}(429x^6y(3y^{12}+277y^{11}+11385y^{10}+274565y^9+4311219y^8+46233051y^7+345455835y^6+1801484135y^5+6462289278y^4+15425254372y^3+23025740280y^2+19051041600y+6486480000)+143x^5y(35y^{12}+3320y^{11}+140575y^{10}+3502744y^9+56996385y^8+635228040y^7+4945087525y^6+26911517512y^5+100775329640y^4+250710484640y^3+388613302800y^2+331987302144y+115857181440)+143x^4y(28y^{12}+2723y^{11}+118598y^{10}+3051283y^9+51485814y^8+597865629y^7+4874144954y^6+27920923009y^5+110537453638y^4+291396674348y^3+477819348648y^2+429022257408y+155658222720)+52x^3y(18y^{12}+1792y^{11}+80193y^{10}+2129237y^9+37275414y^8+451993146y^7+3878017209y^6+23598540581y^5+100326193638y^4+287268980812y^3+516037321848y^2+505974834432y+197017591680)+13x^2y(y+13)^2(5y^{10}+379y^9+12684y^8+246618y^7+3081621y^6+25818279y^5+146545426y^4+554305412y^3+1326486984y^2+1771869312y+876113280)+xy(y^{12}+104y^{11}+4901y^{10}+138424y^9+2611323y^8+34700952y^7+333710663y^6+2347743112y^5+12061579816y^4+44601786944y^3+115119818736y^2+195869441664y+190060335360)+6227020800)$

15	$\frac{1}{14!}(429x^7y(y^{13} + 105y^{12} + 4949y^{11} + 138285y^{10} + 2548203y^9 + 32591475y^8 + 296524247y^7 + 1934641695y^6 + 9006146296y^5 + 29371081320y^4 + 64761694704y^3 + 90607371120y^2 + 71230233600y + 23351328000) + 1001x^6y(7y^{13} + 753y^{12} + 36427y^{11} + 1046565y^{10} + 19863301y^9 + 262055739y^8 + 2462092841y^7 + 16597151175y^6 + 79809813392y^5 + 268535998008y^4 + 609536264432y^3 + 875078262960y^2 + 703024588800y + 234378144000) + 2002x^5y(7y^{13} + 770y^{12} + 38180y^{11} + 1127200y^{10} + 22043598y^9 + 300489780y^8 + 2925005120y^7 + 20478207400y^6 + 102449910883y^5 + 358850685770y^4 + 847029809700y^3 + 1260684703800y^2 + 1044748590912y + 356860183680) + 182x^4y(42y^{13} + 4718y^{12} + 239547y^{11} + 7264168y^{10} + 146429811y^9 + 2065722384y^8 + 20902177461y^7 + 152846077384y^6 + 802538235807y^5 + 2962754475098y^4 + 7388729409492y^3 + 11606659643448y^2 + 10095124659840y + 3583107964800) + 91x^3y(15y^{13} + 1719y^{12} + 89303y^{11} + 2780643y^{10} + 57796585y^9 + 845008197y^8 + 8915766189y^7 + 68487931809y^6 + 381120718660y^5 + 1506557423184y^4 + 4066972726208y^3 + 6974003577648y^2 + 6607869863040y + 2515040236800) + 7x^2y(11y^{13} + 1285y^{12} + 68263y^{11} + 2181881y^{10} + 46776873y^9 + 709649655y^8 + 7829433469y^7 + 63520116803y^6 + 378307951736y^5 + 1629608969560y^4 + 4915343944368y^3 + 9750352815216y^2 + 11137806737280y + 4995141177600) + xy(y^{13} + 119y^{12} + 6461y^{11} + 211939y^{10} + 4687683y^9 + 73870797y^8 + 854224943y^7 + 7353403057y^6 + 47277726496y^5 + 225525484184y^4 + 784146622896y^3 + 1922666722704y^2 + 3134328981120y + 2944310342400) + 87178291200)$

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